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AN ALGEBRAIC STRESS AND HEAT-FLUX MODEL FOR TURBULENT SHEAR FLOW WITH STREAMLINE CURVATURE

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Abstract—The transport equations for Reynolds stress and heat flux in turbulent flow with streamline curvature are closed by modelling the redistributive terms in a manner which reflects the modification of the fluctuating pressure field by the presence of a wall. A set of algebraic equations is derived for thin shear layers with moderate curvature from which the effects of curvature on the mixing-length and the turbulent Prandtl number are deduced. Calculations show that heat transfer is appreciably less affected by streamline curvature than is shear stress.

NOMENCLATURE

$C_1, C_2, \ldots,$ constants in turbulence model;
k, turbulence kinetic energy;
l, turbulence length scale or mixing length;
l_0, mixing length in plane flow;
n, unit vector;
P, production rate of turbulence energy;
P_{ij}, production rate of Reynolds stress $\overline{u_i u_j}$;
$P_{i\gamma}$, part of production rate of heat flux containing mean strain;
p, fluctuating part of the pressure;
r, radius of curvature of streamlines or position vector;
S, curvature parameter;
t, time;
U, circumferential mean velocity component;
u, v, w, fluctuating velocity components;
x, y, z, spatial co-ordinates.

Greek symbols

α, coefficient in mixing-length modification for curvature;
β, eddy-viscosity coefficient;
Γ, mean temperature;
γ, fluctuating temperature;
δ, coefficient in algebraic stress model;
ε, turbulence-energy dissipation rate;
ε_γ, temperature-variance dissipation rate;
η, eddy-diffusivity coefficient;
σ_t, turbulent Prandtl number;
ϕ, pressure-rate-of-strain or pressure-temperature gradient correlations in Reynolds stress and heat-flux equations, or algebraic model coefficients.

1. INTRODUCTION

THE TURBULENCE in thin shear layers is known to be highly sensitive to streamline curvature in the plane of the mean shear. Turbulent transport of heat and momentum is reduced by curvature when the angular momentum of the flow increases in the direction of the radius of curvature and is increased when the angular momentum decreases with radius. In contrast to laminar flow, where the fractional change due to curvature in, for example, the shear stress is of the same order as the ratio of the shear layer thickness to the radius, turbulent flow measurements show fractional changes an order of magnitude greater. The effects of curvature on a thin shear layer in turbulent flow can therefore be significant for radii of order one hundred times the shear layer thickness. An extreme case is the observed extinction of the turbulent shear stress in flow over a strongly curved convex surface. These effects are important in many flows of engineering interest. A good example is the turbomachine blade where the effects of curvature may reduce the skin friction and heat transfer on the convex surface by as much as ten or twenty per cent in a typical case and increase these quantities by a similar amount on the concave surface.

Nearly all attempts to account for streamline curvature effects in calculation schemes have involved empirical modifications to eddy-viscosity formulae developed for flows with insignificant curvature. In particular, many workers have used Bradshaw's [1] modification for the mixing length, l:

$$l = l_0(1-\alpha S)/(1-S) \quad (1)$$

where l_0 is the mixing length appropriate to plane flow and S is a curvature parameter defined as:

$$S \equiv \frac{U/r}{\partial U/\partial z}. \quad (2)$$

Here U is the mean circumferential velocity component, r the radius of curvature and z the co-ordinate distance normal to the streamlines. α in equation (1) is an empirical constant of order 10. This treatment has proved to be reasonably successful (see Foloyan and Whitelaw [2] for a useful short survey of recent applications) but it is limited by the need to prescribe not only l_0 but also α differently for different flows [1].

This difficulty is avoided in the two-equation closure proposed by Launder *et al.* [3] where a length-scale distribution is obtained from the solution of modelled equations for the turbulent kinetic energy and its dissipation rate. The energy production rate due to extra strain associated with curvature appears in exact form in the energy equation, but curvature effects have to be modelled empirically in the dissipation equation.

The development of multi-equation models may have been inhibited by the view that there is insufficient experimental evidence to justify closures which are significantly more complex than those at eddy-viscosity level. On the other hand it can be argued that low-level closures cannot adequately account for all the effects of extra strain in complex shear layers.

There is a fairly close analogy between the effects on thin shear layers of streamline curvature and buoyancy [4]. Multi-equation models have been widely used (e.g. [5–7]) for calculating buoyancy-affected turbulence since Reynolds stresses and heat fluxes are affected by buoyant production in a fairly complicated way not easily accounted for by a simple eddy-viscosity formulation. Mellor [8] has in fact used the same stress-closure approximations for both types of flow to produce, for curved flow, a function modifying the eddy viscosity. Irwin and Smith [9] simplified the stress-closure of Launder *et al.* [10] to calculate the development of boundary layers and wall jets on curved surfaces with generally good agreement with experiment. The most important result to emerge from this study was that the observed curvature effects could be accounted for by the relatively small production terms appearing in the individual Reynolds-stress equations.

Scarcely any attention has been given to the problem of heat transfer in flows affected by curvature. In the absence of experimental evidence to the contrary a constant turbulent Prandtl number has usually been assumed for eddy-viscosity models. Now in stratified flow the turbulent Prandtl number is found to be strongly affected by buoyancy and there is every reason to expect a similar dependence on curvature. The direct application of buoyant-flow data would probably put too large a strain on the buoyancy/curvature analogy and it is only by closing the turbulent heat-flux equations that the behaviour of the turbulent Prandtl number can be predicted. Turbulence modelling at this level of closure seems not to have been attempted for curved flow; in Mellor's [8] treatment only the stress equations of the stratified-flow model were used for curved flow.

The present study has been prompted by the need for accurate prediction methods for flow over curved surfaces and the realisation that the effects of curvature on heat transfer could only be accounted for by modelling the Reynolds-stress and the heat-flux equations. The model used is that originally developed for buoyancy-affected turbulence by Gibson and Launder [11, 12] and the approximations made to close the Reynolds stress equations are similar to those of [9] and [10] for free shear flows. The influence of the wall on the fluctuating pressure field is modelled rather differently and although the wall-damping effects are still expressed through a length-scale function this is modified as in equation (1) to take account of curvature. Studies of stratified wall flow [12] showed that this modification was essential if the effects of buoyancy on the normal stress ratios were to be accurately predicted and there is no reason to suppose that the turbulence in curved flow will respond very differently. The heat-flux equations are treated in a precisely parallel way with appropriate modifications for wall damping. Use of the algebraic stress and flux modelling technique enables the turbulent Prandtl number and the heat-flux correlation coefficient to be expressed in terms of the curvature parameter S. It is found that the turbulent Prandtl number decreases with increasing S, that is with increasing convexity. This implies that turbulent heat transfer is less affected by curvature than is the shear stress, a result which contrasts sharply with predictions and data for stratified flow in which the turbulent Prandtl number increases with increasing stability.

2. TURBULENCE MODEL FOR BOUNDARY-LAYER FLOWS WITH SMALL CURVATURE

2.1. *The Reynolds stress equations*

Consideration is restricted to two-dimensional incompressible flows in which the curvature of the mean streamlines is small. Transport equations for the Reynolds stresses are obtained from the Navier–Stokes equations as described in [1]. For boundary-layer flow at Reynolds numbers high enough for the small scale motion to be assumed isotropic the Reynolds stress equations are [9]:

$$\frac{D\overline{u^2}}{Dt} = -\frac{\partial \overline{wu^2}}{\partial z} + \frac{2}{1+z/r}\overline{\frac{p}{\rho}\frac{\partial u}{\partial x}} - 2\left[\overline{u^2}\frac{\partial U}{\partial x} + \overline{uw}\left(\frac{\partial U}{\partial z} + \frac{U}{r}\right)\right] - \tfrac{2}{3}\varepsilon \quad (3)$$

$$\frac{D\overline{v^2}}{Dt} = -\frac{\partial \overline{wv^2}}{\partial z} + 2\overline{\frac{p}{\rho}\frac{\partial v}{\partial y}} - \tfrac{2}{3}\varepsilon \quad (4)$$

$$\frac{D\overline{w^2}}{Dt} = -\frac{\partial}{\partial z}\left(\overline{w^3} + \frac{2\overline{pw}}{\rho}\right) + 2\overline{\frac{p}{\rho}\frac{\partial w}{\partial z}} + 2\left(\overline{w^2}\frac{\partial U}{\partial x} + 2\overline{uw}\frac{U}{r}\right) - \tfrac{2}{3}\varepsilon \quad (5)$$

$$\frac{D\overline{uw}}{Dt} = -\frac{\partial}{\partial z}\left(\overline{uw^2} + \frac{\overline{pu}}{\rho}\right) + \overline{\frac{p}{\rho}\left(\frac{1}{1+z/r}\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)} - \left[\overline{w^2}\frac{\partial U}{\partial z} - (2\overline{u^2} - \overline{w^2})\frac{U}{r}\right] \quad (6)$$

where ε is the turbulent-energy dissipation rate, p is the fluctuating part of the pressure, ρ is the density and:

$$\frac{\mathrm{D}}{\mathrm{D}t} \equiv U\frac{\partial}{\partial x} + W\frac{\partial}{\partial z}.$$

The transport equation for turbulent energy, k, is obtained by adding equations (3), (4) and (5):

$$\frac{\mathrm{D}k}{\mathrm{D}t} = -\frac{\partial}{\partial z}\left(\overline{wk'} + \frac{\overline{pw}}{\rho}\right) - \left[(\overline{u^2} - \overline{w^2})\frac{\partial U}{\partial x} + \overline{uw}\left(\frac{\partial U}{\partial z} - \frac{U}{r}\right)\right] - \varepsilon \quad (7)$$

where k' is the fluctuating part of k. Terms involving the longitudinal acceleration $\partial U/\partial x$ contribute to the production of $\overline{u^2}$, $\overline{w^2}$ and k. These terms are often ignored in modelling plane flows but, since they are of the same order as the production terms involving U/r, they ought logically to be included when the effects of curvature are considered. However, since it seems improbable that the two effects are significantly coupled, at any rate for mild curvature, and the study of Irwin and Smith [9] suggested that the turbulence structure is much less sensitive to the effects of acceleration than to those of curvature, these terms will be discarded in the present study as they were in [9].

To convert equations (3)–(6), the mean-flow momentum equation, and the continuity equations into a closed set for U, W and the Reynolds stresses, the turbulence quantities on the RHS of (3)–(6) must be expressed in terms of the mean velocities and the Reynolds stresses. The main obstacle to closure is presented by the pressure–strain correlation appearing as the second term in each of the stress equations. It can be shown (e.g. [10]) that two kinds of interaction give rise to these correlations: one involving only turbulence quantities and another arising from the presence of the mean rate of strain. If these terms are denoted by ϕ_{ij} and tensor notation is used for brevity:

$$\phi_{ij} = \phi_{ij,1} + \phi_{ij,2} \quad (8)$$

where

$$\phi_{11} \equiv \frac{2}{1+z/r}\,\overline{\frac{p}{\rho}\frac{\partial u}{\partial x}}\,;\; \phi_{22} \equiv 2\,\overline{\frac{p}{\rho}\frac{\partial v}{\partial y}}\,;\; \phi_{33} \equiv 2\,\overline{\frac{p}{\rho}\frac{\partial w}{\partial z}}.$$

The two components of ϕ_{ij} are modelled as in [11] and [12]:

$$\phi_{ij,1} = -C_1\frac{\varepsilon}{k}(\overline{u_i u_j} - \tfrac{2}{3}\delta_{ij}k) \quad (9)$$

$$\phi_{ij,2} = -C_2(P_{ij} - \tfrac{2}{3}\delta_{ij}P) \quad (10)$$

where P_{ij} is the production rate of $\overline{u_i u_j}$ which appears as the third term on the RHS of equations (3)–(6) and is zero in the $\overline{v^2}$ equation (4). Thus, in the equation for $\overline{u^2}$, with the term containing $\partial U/\partial x$ neglected:

$$P_{11} = -2\overline{uw}\left(\frac{\partial U}{\partial z} + \frac{U}{r}\right).$$

P is the production rate of turbulent energy which, when the acceleration term is again omitted, is seen from equation (8) to be:

$$P = -\overline{uw}\left(\frac{\partial U}{\partial z} - \frac{U}{r}\right) = -\overline{uw}\frac{\partial U}{\partial z}(1-S). \quad (11)$$

C_1 and C_2 are constants for high-Reynolds-number turbulence which are determined by reference to plane-flow data.

In a simple shear flow the proximity of a rigid wall modifies the fluctuating pressure field so as to impede the transfer of energy from the streamwise direction to that normal to the wall. The relative magnitude of the shear stress is also reduced. Shir [13] proposed the following addition to $\phi_{ij,1}$ to account for near-wall effects:

$$\phi'_{ij,1} = C'_1\frac{\varepsilon}{k}(\overline{u_k u_m}n_k n_m\delta_{ij} - \tfrac{3}{2}\overline{u_k u_i}n_k n_j - \tfrac{3}{2}\overline{u_k u_j}n_k n_i)f\left(\frac{l}{n_i r_i}\right). \quad (12)$$

n_i is the unit vector normal to the surface, r_i is the position vector (not to be confused with radius of curvature) and l is some suitable turbulence length scale. In [12] the idea expressed by equation (12) was applied also to the mean-strain component, $\phi_{ij,2}$ as:

$$\phi'_{ij,2} = C'_2(\phi_{km,2}n_k n_m\delta_{ij} - \tfrac{3}{2}\phi_{ik,2}n_k n_j - \tfrac{3}{2}\phi_{jk,2}n_k n_i)f\left(\frac{l}{n_i r_i}\right) \quad (13)$$

where C'_1 and C'_2 are constants chosen to give the right stress levels in the uniform-stress layer close to a plane wall.

The transport terms are treated by the technique known as algebraic stress modelling [14]. The net transport of $\overline{u_i u_j}$ is assumed to be proportional to the net transport of k multiplied by the factor $\overline{u_i u_j}/k$. Thus:

$$\frac{\mathrm{D}\overline{u_i u_j}}{\mathrm{D}t} - \mathscr{D}(\overline{u_i u_j}) = \frac{\overline{u_i u_j}}{k}\left\{\frac{\mathrm{D}k}{\mathrm{D}t} - \mathscr{D}(k)\right\} = \frac{\overline{u_i u_j}}{k}(P-\varepsilon) \quad (14)$$

where $\mathscr{D}(\overline{u_i u_j})$ denotes diffusion of $\overline{u_i u_j}$:

$$\mathscr{D}(\overline{u_i u_j}) \equiv \frac{\partial}{\partial x_k}\left[\overline{u_i u_j u_k} + \overline{\frac{p}{\rho}(\delta_{jk}u_i + \delta_{ik}u_j)}\right]$$

and

$$\mathscr{D}(k) \equiv \frac{1}{2}\frac{\partial}{\partial x_k}\left(\overline{u_i u_i u_k} + \delta_{ik}\overline{\frac{p}{\rho}u_i}\right).$$

This treatment appears to be reasonably accurate for thin shear layers except near an axis of symmetry.

Substitution of the model assumptions yields the following set of algebraic equations for the Reynolds stresses:

$$\frac{\overline{u^2}}{k} = \frac{2}{3}\left(1 + \frac{P}{\varepsilon}\phi_4\right) + \phi_5\frac{\overline{w^2}}{k} + \frac{2S}{1-S}\phi_3\frac{P}{\varepsilon} \quad (15)$$

$$\frac{\overline{v^2}}{k} = \frac{2}{3}\left(1-\frac{P}{\varepsilon}\phi_1\right)+\phi_5\frac{\overline{w^2}}{k}+\frac{2S}{1-S}\phi_6\frac{P}{\varepsilon} \quad (16)$$

$$\frac{\overline{w^2}}{k} = \frac{2}{3}\frac{1-\frac{P}{\varepsilon}\phi_2}{1+2\phi_5}-\frac{2S}{1-S}\frac{\phi_7}{1+2\phi_5}\cdot\frac{P}{\varepsilon} \quad (17)$$

$$\frac{-\overline{uw}}{k} = \left(\frac{\beta}{1-S}\cdot\frac{P}{\varepsilon}\cdot\frac{\overline{w^2}}{k}\right)^{0.5} \quad (18)$$

$$\overline{uw} = -\beta\frac{k}{\varepsilon}\overline{w^2}\frac{\partial U}{\partial z}. \quad (19)$$

The coefficients ϕ are functions of the model constants. The length scale function appearing in equations (12) and (13) is written simply as f and:

$$\phi_1 = (1-C_2-C_2'C_2f)/\delta \quad (20)$$

$$\phi_2 = (1-C_2+2C_2'C_2f)/\delta \quad (21)$$

$$\phi_3 = (2-2C_2+2C_2'C_2f)/\delta \quad (22)$$

$$\phi_4 = (2-2C_2+C_2'C_2f)/\delta \quad (23)$$

$$\phi_5 = C_1'f/\delta \quad (24)$$

$$\phi_6 = 2C_2'C_2f/\delta \quad (25)$$

$$\phi_7 = (2-2C_2+4C_2'C_2f)/\delta \quad (26)$$

$$\beta = \phi\left\{1-\left(2\frac{\overline{u^2}}{\overline{w^2}}-1\right)S\right\} \quad (27)$$

$$\phi = (1-C_2+1.5C_2'C_2f)/(\delta+1.5C_1'f) \quad (28)$$

$$\delta = C_1+\frac{P}{\varepsilon}-1. \quad (29)$$

The form of the wall-damping function $f(l/z)$ must now be specified. This quantity must of course vanish in turbulence unaffected by the presence of a wall, when l/z is zero. It is assumed, as in earlier work [10, 12], that f is directly proportional to l/z, and it is convenient to choose the constant of proportionality such that the function is unity in near-wall turbulence. A suitable length scale characteristic of the energy-containing motion may be formed from the shear stress and dissipation rate as:

$$l \equiv \frac{(-\overline{uw})^{3/2}}{\varepsilon}. \quad (30)$$

In plane flow this is the mixing length which is equal to kz near a plane wall. Thus f may be defined by:

$$f = \frac{l}{kz} \quad (31)$$

so that, for plane flow subjected only to the simple strain $\partial U/\partial z$, f is unity close to a wall and approaches zero in the outer region.

Now while the extra strain associated with mild streamwise curvature alters the production rate of turbulent energy, equation (11), by the factor $(1-S)$, the observed effects on the shear stress are much greater. According to Bradshaw [1] the change is what would be predicted by a simple turbulence model if the production terms had changed, not by the factor $(1-S)$ but by a factor $(1-\alpha S)$, where α varies from case to case but is always of order 10. Examination of the production terms in the shear-stress equation (6), and the form of the coefficient β arising from the model approximations, shows this factor to be:

$$1-\left(\frac{2\overline{u^2}}{\overline{w^2}}-1\right)S.$$

Bradshaw's analysis for local-equilibrium turbulence shows how the effect on the length scale may be accounted for by multiplying the mixing-length distribution for plane flow by the factor $(1-\alpha S)/(1-S)$. Combination of this result with equation (31) produces for the wall-damping function:

$$f = \frac{1-\alpha S}{1-S} \quad (32)$$

in which:

$$\alpha \equiv 2\frac{\overline{u^2}}{\overline{w^2}}-1. \quad (33)$$

The modelled equations contain four empirical constants which are assigned the values determined in [12] from plane-flow data: $(C_1, C_2, C_1', C_2') = (1.8, 0.6, 0.5, 0.3)$. The stress levels then obtained for local-equilibrium turbulence ($P = \varepsilon$) in plane flows are shown in Table 1. For free flow with $f = 0$ these correspond closely to the measurements of Champagne *et al.* [15] in a plane homogeneous shear layer and, for $f = 1$, to the consensus of near-wall turbulence data cited in [10].

Table 1. Comparison of measured and calculated Reynolds stresses in plane equilibrium shear flows, $P/\varepsilon = 1$

	$\overline{u^2}/k$	$\overline{v^2}/k$	$\overline{w^2}/k$	$-\overline{uw}/k$
Plane homogeneous shear layer				
experimental data [15]	0.97	0.54	0.49	0.33
model results	0.96	0 52	0.52	0.34
Near-wall turbulence				
a consensus of data [10]	1.17	0.59	0.24	0.24
model results	1.10	0.65	0.25	0.26

The turbulence model developed so far consists of a set of algebraic equations (15)–(17) for the normal stress components $\overline{u_iu_i}/k$, and a gradient–diffusion expression (19) for the shear stress. These equations contain two scalar properties of turbulence as unknowns: k and ε. The former can be obtained by solving the differential transport equation (7) using the turbulent viscosity implied in (19) to model the diffusion term. The dissipation rate can also be found, in principle, from the solution of an approximated transport equation. Alternatively, equation (30) may be used to determine ε from a specified length-scale distribution modified for curvature effects as suggested by (32).

In the present study, however, attention is confined to the algebraic equations. The turbulent energy and dissipation rate remain as unknowns, k/ε

defining a characteristic time scale, and there is no need to specify the length scale since l is used only to define the wall damping function which is given explicitly by equation (32). The calculations are restricted to local-equilibrium turbulence for which P/ε is unity.

2.2. *The heat-flux equations*

The transport equations for the heat fluxes are obtained by multiplying the equation for the instantaneous temperature $(\Gamma+\gamma)$ by u_i and that for (U_i+u_i) by γ. The resulting equations are added and averaged to yield equations for $\overline{u_i\gamma}$ which, for a thin shear layer and to the same level of approximation implied in (3)–(6), are written:

$$\frac{D\overline{u\gamma}}{Dt} = -\frac{\partial}{\partial z}\overline{uw\gamma} + \frac{1}{1+z/r}\overline{\frac{p}{\rho}\frac{\partial\gamma}{\partial x}} - \left[\overline{u\gamma}\frac{\partial U}{\partial x} + \overline{w\gamma}\left(\frac{\partial U}{\partial z}+\frac{U}{r}\right) + \overline{uw}\frac{\partial\Gamma}{\partial z}\right] \quad (34)$$

$$\frac{D\overline{w\gamma}}{Dt} = -\frac{\partial}{\partial z}\left(\overline{w^2\gamma} + \frac{\overline{p\gamma}}{\rho}\right) + \overline{\frac{p}{\rho}\frac{\partial\gamma}{\partial z}} + \left(2\overline{u\gamma}\frac{U}{r} - \overline{w^2}\frac{\partial\Gamma}{\partial z}\right). \quad (35)$$

The fluctuating pressure–temperature-gradient correlations appearing as the second term on the RHS of each equation are modelled in an exactly parallel way to the equivalent terms in the stress equations. Each can be shown [16] to consist of two components, one involving only turbulence correlations and the other containing the mean rate of strain. These are denoted by $\phi_{i\gamma}$ and modelled as in [11, 12] as:

$$\phi_{i\gamma} = \phi_{i\gamma,1} + \phi_{i\gamma,2} \quad (36)$$

$$\phi_{i\gamma,1} = -C_{1\gamma}\frac{\varepsilon}{k}\overline{u_i\gamma} \quad (37)$$

$$\phi_{i\gamma,2} = -C_{2\gamma}P_{i\gamma} \quad (38)$$

where $P_{i\gamma}$ is that part of the production rate of $\overline{u_i\gamma}$ which contains a component of mean strain. If the effects of longitudinal acceleration are again ignored, $P_{u\gamma} \equiv -\overline{w\gamma}(\partial U/\partial z + U/r)$ and $P_{w\gamma} \equiv 2\overline{u\gamma}U/r$.

The near-wall modifications equivalent to equations (12) and (13) are:

$$\phi_{i\gamma,1}' = -C_{1\gamma}'\frac{\varepsilon}{k}\overline{u_k\gamma}\,n_i n_k f\left(\frac{l}{n_i r_i}\right) \quad (39)$$

$$\phi_{i\gamma,2}' = C_{2\gamma}'C_{2\gamma}P_{k\gamma}n_i n_k f\left(\frac{l}{n_i r_i}\right). \quad (40)$$

The "algebraic heat-flux model" corresponding to equation (14) is [11]:

$$\frac{D\overline{u_i\gamma}}{Dt} - \mathscr{D}(\overline{u_i\gamma}) = \frac{\overline{u_i\gamma}}{(k\overline{\gamma^2})^{0.5}}\left[\frac{D}{Dt}(k\overline{\gamma^2})^{0.5} - \mathscr{D}(k\overline{\gamma^2})^{0.5}\right] = \frac{\overline{u_i\gamma}}{2k}(P-\varepsilon) + \frac{\overline{u_i\gamma}}{2\overline{\gamma^2}}(P_\gamma - \varepsilon_\gamma) \quad (41)$$

where P_γ and ε_γ are the rates of production and dissipation of the temperature variance $\overline{\gamma^2}$. These model approximations are used to transform equations (34) and (35) into the following algebraic expressions for the turbulent heat fluxes:

$$\overline{u\gamma} = -\phi_\gamma\frac{k}{\varepsilon}\overline{uw}\frac{\partial\Gamma}{\partial z} - \phi_\gamma'\frac{k}{\varepsilon}\overline{w\gamma}\frac{\partial U}{\partial z} \quad (42)$$

$$\overline{w\gamma} = -\phi_{\gamma 1}\frac{k}{\varepsilon}\overline{w^2}\frac{\partial\Gamma}{\partial z} + \phi_{\gamma 1}'\frac{k}{\varepsilon}\overline{u\gamma}\frac{\partial U}{\partial z} \quad (43)$$

where

$$\phi_\gamma = \left[C_{1\gamma} + \frac{1}{2}\left(\frac{P}{\varepsilon}-1\right) + \frac{1}{2}\frac{k}{\varepsilon}\frac{\varepsilon_\gamma}{\overline{\gamma^2}}\left(\frac{P_\gamma}{\varepsilon_\gamma}-1\right)\right]^{-1} \quad (44)$$

$$\phi_\gamma' = \phi_\gamma(1-C_{2\gamma})(1+S) \quad (45)$$

$$\phi_{\gamma 1} = \phi_\gamma(1+\phi_\gamma C_{1\gamma}'f)^{-1} \quad (46)$$

$$\phi_{\gamma 1}' = 2\phi_{\gamma 1}S(1-C_{2\gamma}+C_{2\gamma}C_{2\gamma}'f). \quad (47)$$

Three of the four constants may be fixed by reference to heat-transfer data from high-Reynolds number turbulence in plane flow. The values recommended by Gibson and Launder [12] are: $(C_{1\gamma}, C_{2\gamma}, C_{1\gamma}') = (3.0, 0.33, 0.5)$.

These were chosen to give values of the turbulent Prandtl number in plane flow of 0.67 and 0.92 for free and near-wall turbulence respectively. It will be noted that wall-suppression effects are absent from the equation for $\overline{u\gamma}$ and for plane flow only $C_{1\gamma}'$ appears in the $\overline{w\gamma}$ equation. The wall modification involving the mean strain rate appears only for flow over a curved surface and the constant $C_{2\gamma}'$ cannot, therefore, be evaluated from plane-flow data. Calculations for curved and buoyant flow show that this term appears to exercise little influence and so, in the absence of more definite information, $C_{2\gamma}'$ has been set equal to zero.

The two heat-flux equations (42) and (43) contain, as additional unknowns, the mean-square temperature fluctuation and dissipation rate. The equation for $\overline{\gamma^2}$ is the simplest of all turbulence transport equations and its solution presents no difficulty provided that ε_γ can be modelled satisfactorily. An exact transport equation for ε_γ can be derived [16] from the Navier–Stokes equation but a satisfactory modelled form has yet to be developed. The usual practice is to express ε_γ in terms of ε through equation (53) below. In the present study of local-equilibrium turbulence the production and dissipation rates of $\overline{\gamma^2}$ are assumed equal so that the term containing $\overline{\gamma^2}$, equation (44), vanishes. $\overline{\gamma^2}$ will, however, be approximated later in order to form a correlation coefficient.

3. RESULTS AND DISCUSSION

3.1. *Effects of curvature on the Reynolds stress*

The extra strain rate U/r associated with streamline curvature contributes to the production of Reynolds stress through the terms appearing in the exact transport equations (3)–(6). When S is positive the direct effect is to diminish the shear stress and turbulence energy relative to the corresponding plane-flow values. In conditions of strong stabilising curvature the shear stress falls to zero at a critical value of S predicted by the model to be 0.17 for the particular case of local equilibrium turbulence. This figure agrees reasonably well with the critical value of 0.15 found by So and Mellor [17] in strongly curved convex wall flow. It is argued that wall-damping effects are negligible for strong convex curvature and the critical value of S is independent of the form of the length scale function f. As was shown by Irwin and Smith [9] the observed behaviour of the shear stress is sensitive to the values of the normal stresses and can be attributed to the extra production terms in the stress equations. For a local equilibrium flow unaffected by the presence of a wall the model predicts a fall in the normal stress ratio $\overline{w^2}/\overline{u^2}$ from 0.54 with zero curvature, $S = 0$, to 0.29 at the critical $S = 0.17$. The predicted variation in free flow of the normal stress ratio and $\overline{uw}/k$ with S is shown in Fig. 1.

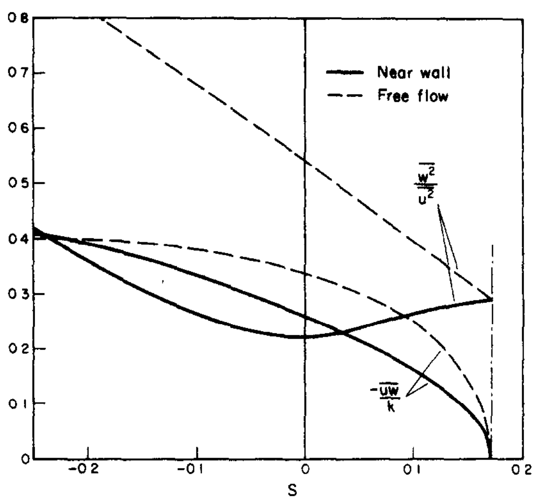

FIG. 1. Predicted variation with S of the shear-stress and normal-stress ratios for local equilibrium turbulence.

The predicted behaviour of the normal-stress ratio in convex wall flow, also shown in Fig. 1, is strikingly different although, not surprisingly, it is similar to that in stably-stratified flow [12]. It must first be observed that one effect of wall damping in plane flow is to depress $\overline{w^2}$ relative to $\overline{u^2}$ so that the ratio is only about one half that in a plane homogeneous shear layer, (Table 1). Now in flow over a convex wall the normal stresses are subjected to opposing influences. The direct effect of curvature, expressed through the production terms, is to diminish the stress ratio as in free curved flow. At the same time the wall-damping effect itself is reduced by convex curvature so that the normal stresses tend to revert to their free-flow values. This is what is implied by the near-wall modifications to the pressure-strain terms in equations (12) and (13), and the observed behaviour of the length scale, equation (1). The model predictions for near-wall turbulence show $\overline{w^2}/\overline{u^2}$ increasing with increasing positive S to the free flow value of 0.29 at the critical curvature where the shear stress collapses.

The predicted variation with S of f and β is shown in Fig. 2. The coefficient α in the length scale function, given by equation (33), varies from 7.87 for small curvature to 5.85 at the critical condition of vanishing shear stress. These values may be compared with Bradshaw's [1] recommendations for wall flows which range from $\alpha = 6$ for a wall jet on a convex surface to $\alpha = 14$ for a boundary layer on a convex surface. The comparison is not an exact one since the data examined by Bradshaw were not extensive enough to warrant correlations of α as functions of distance from the wall. These recommendations must be treated as global values in contrast to the results of the present study of the near-wall region. Additional support is provided by data from buoyancy-affected turbulence although it is probably unwise to rely too heavily on the buoyancy-curvature analogy. α values in the range 7–10 fit the wind-tunnel measurements of Arya and Plate [18] while the atmospheric surface-layer data

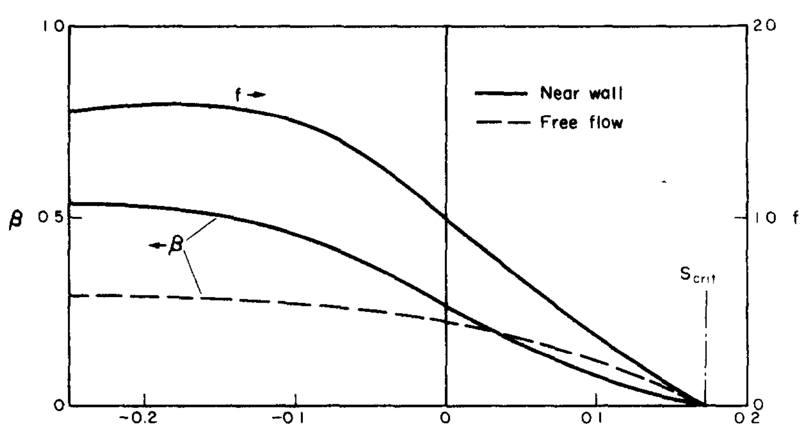

FIG. 2. Predicted variation with S of the length scale function f and the eddy-viscosity coefficient β for local equilibrium turbulence.

reviewed by Busch [19] suggest values ranging from 4.7 to 7.0.

The convex-wall measurements of So and Mellor [17] may be cited in at least partial support of the predicted behaviour of the normal-stress ratio. From the initial, flat, approach flow to the point of vanishing shear stress the measured $\overline{w^2}/k$ in the near-wall region increased by roughly 30% and $\overline{w^2}/\overline{u^2}$ by roughly 70%. On the other hand the measurements show $\overline{u^2}/k$ decreasing from about unity to about 0.8, a trend not predicted by the model which gives this quantity as roughly constant for positive S.

3.2. *Heat transfer*

The algebraic heat-flux equations (42) and (43) can be combined to give a gradient-transfer expression for the cross-stream heat flux:

$$\overline{w\gamma} = -\eta \frac{k}{\varepsilon}\overline{w^2}\frac{\partial \Gamma}{\partial z} \tag{48}$$

from which, using the parallel expression (19) for the shear stress, the turbulent Prandtl number can be obtained as:

$$\sigma_t = \frac{\beta}{\eta} = \frac{\beta + a\phi'_{\gamma}\phi'_{\gamma 1}}{\phi_{\gamma 1} - a\phi_{\gamma}\phi'_{\gamma 1}} \tag{49}$$

where

$$a \equiv \frac{k}{\overline{w^2}}\frac{P/\varepsilon}{1-S}. \tag{50}$$

The predicted variation of η and σ_t with S for local equilibrium turbulence is plotted in Fig. 3. The eddy-

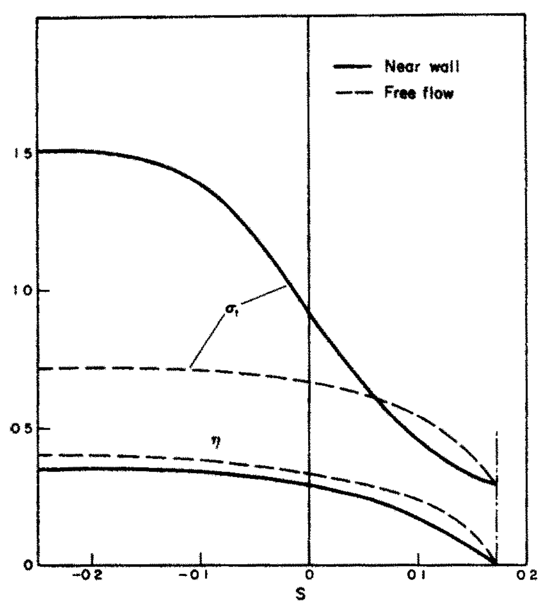

FIG. 3. Predicted variation with S of the eddy-diffusivity coefficient η and the turbulent Prandtl number σ_t for local-equilibrium turbulence.

diffusivity coefficient η appears to be virtually unaffected by destabilising curvature ($S < 0$) but falls with positive increasing S to zero at the critical curvature where the shear stress vanishes. The decrease in turbulent Prandtl number with increasing S reflects, particularly for wall-flow, the parallel behaviour of the eddy-viscosity coefficient β. The implication is that the heat flux is significantly less affected by streamline curvature than is shear stress, as may be deduced from the relative magnitude of the extra-strain production terms in the transport equations for $\overline{uw}$ and $\overline{w\gamma}$. The factor multiplying the plane-flow production rate of heat flux reduces to, after some manipulation:

$$1 - \frac{2}{\sigma_t}\frac{\overline{u\gamma}}{\overline{w\gamma}}\frac{\overline{uw}}{\overline{w^2}}S.$$

When the plane wall-flow data used to establish the model constants are substituted the coefficient on S turns out to be about 4.6, or only about 60% of the value of α in the factor $(1-\alpha S)$ multiplying the shear-stress production rate. In buoyancy-affected turbulence the opposite situation arises. The experimental data and the model predictions of Gibson and Launder [11, 12] show that σ_t invariably *increases* with increasing stability. Buoyancy production of heat flux is relatively greater than that of shear stress although the effect is not so easily seen because the coupling between the transport equations for buoyant flow is to diminish the tendency of σ_t to increase with increasing stability to the extent that it becomes approximately constant in strongly-stable conditions [12, 19].

Some further manipulation of the model equations produces an expression for the cross-stream/streamwise heat-flux ratio:

$$\frac{\overline{w\gamma}}{\overline{u\gamma}} = \frac{\overline{w^2}}{\overline{uw}}\frac{\beta}{\phi_{\gamma}\sigma_t + \phi'_{\gamma}}. \tag{51}$$

The model predicts that this quantity decreases with increasing S as is shown in Fig. 4. Also plotted in the

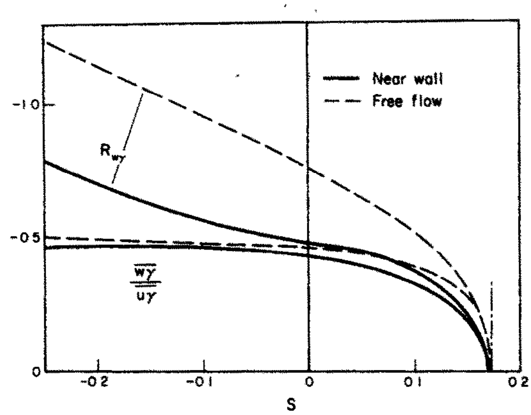

FIG. 4. Predicted variation with S of the heat-flux ratio and correlation coefficient for local equilibrium turbulence.

figure is the heat-flux correlation coefficient which is obtainable as a result of further assumptions regarding the temperature variance $\overline{\gamma^2}$. If it is assumed for flows in local equilibrium that the production and dissipation rates of $\overline{\gamma^2}$ also balance, i.e. that $P_{\gamma} = \varepsilon_{\gamma}$, the transport equation for $\overline{\gamma^2}$ reduces to:

$$\overline{\gamma^2} = -C_{\gamma}\frac{k}{\varepsilon}\overline{w\gamma}\frac{\partial\Gamma}{\partial z} \quad (52)$$

where C_{γ} is twice the ratio of the time scales for temperature and velocity fluctuations:

$$C_{\gamma} \equiv \frac{\varepsilon}{\varepsilon_{\gamma}}\frac{\overline{\gamma^2}}{k}. \quad (53)$$

The value of C_{γ} is somewhat controversial at present but here, in accordance with earlier model studies [11, 12], it has been taken as equal to 1.6. A little algebra produces the correlation coefficient as:

$$R_{w\gamma} \equiv \frac{-\overline{w\gamma}}{(\overline{w^2})^{1/2}(\overline{\gamma^2})^{1/2}} = \left(\frac{\eta}{C_{\gamma}}\right)^{1/2} \quad (54)$$

which is plotted against S in Fig. 4. The effect of curvature on C_{γ} is unknown and cannot be predicted by the model.

There seem to be no experimental data against which these conjectures may be tested. The supersonic-flow measurements of Thomann [20] showed that heat-transfer rates increased relative to flat-plate values on a concave surface and decreased on a convex surface. Since no skin-friction measurements were made the relative effects of curvature could not be ascertained.

4. CONCLUDING REMARKS

The model study suggests that the effects of streamline curvature on momentum and heat transfer are mainly due to the extra production terms appearing in the transport equations for Reynolds stress and heat-flux. It is further suggested that the influence of a solid wall on the turbulence is itself modified by curvature and that the effects of curvature on the turbulence structure, as expressed by the normal-stress ratio, differ significantly between free and near-wall flows. The first conclusion was reached by Irwin and Smith [9]; the second is supported by the observed and predicted behaviour of stably-stratified buoyant flow [12] with which there is some analogy.

The model produces an explicit form, equation (32), for the empirical, Monin-type, modification of the mixing length for curvature. Values of the coefficient α, which depends upon the normal stress ratio, fall within the range recommended and used by other investigators [1].

It has been shown that the effects of streamline curvature on heat-transfer are probably significantly less than on shear stress. Thus the proportionate reduction of heat transfer on, for example, the convex surface of a turbine blade, may be expected to be less than that for skin friction. This is an important result which suggests that the usual practice of using a constant turbulent Prandtl number in prediction methods may well provide misleading estimates of the heat transfer from curved surfaces.

The study also suggests a strategy for calculation methods and the type of model most suited for the prediction of curved flow. There seems to be no justification for full second-order methods, i.e. those involving numerical solution of the stress and heat flux equations. Two equation closures necessarily involve ad hoc modifications to the length-scale equations or its equivalent to take account of curvature. The present analysis can, however, conveniently be used in a one-equation scheme. There is no difficulty involved in solving the turbulence energy equation for thin curved shear layers. The Reynolds normal stresses are then available from algebraic relationships, the length scale can be found and the energy dissipation rate calculated. The shear stress and heat flux are then calculated from gradient-transfer formulae in which the coefficients are functions of the degree of curvature and the variation of the turbulent Prandtl number is taken into account. Such a scheme would seem at the present state of the art to afford the best prospects for calculations.

Attention is drawn to the lack of experimental data against which the conjectures of this study may be tested. There is an urgent need for more information in this field, particularly from flows with heat transfer.

REFERENCES

1. P. Bradshaw, Effect of streamwise curvature on turbulent flows, AGARDograph No. 169 (1973).
2. C. O. Foloyan and J. H. Whitelaw, The effectiveness of two-dimensional film-cooling over curved surfaces, ASME paper 76-HT-31 (1976).
3. B. E. Launder, C. H. Priddin and B. S. Sharma, The calculation of turbulent boundary layers on spinning and curved surfaces, *J. Fluid Engng* **99I**, 231–239 (1977).
4. P. Bradshaw, The analogy between streamline curvature and buoyancy in turbulent shear flow, *J. Fluid Mech.* **36**(1), 177–191 (1969).
5. J. L. Lumley, A model for computation of stratified turbulent flows, in *Proceedings of the International Symposium on Stratified Flow*, Novosibirsk (1972).
6. G. L. Mellor, Analytic prediction of the properties of stratified planetary surface layers, *J. Atmos. Sci.* **30**, 1061–1069 (1973).
7. J. C. Wyngaard and O. R. Cote, The evolution of a convective planetary boundary layer—a higher-order-closure model study, *Boundary-Layer Met.* **7**, 289–308 (1974).
8. G. L. Mellor, A comparative study of curved flow and density-stratified flow, *J. Atmos. Sci.* **32**, 1278–1282 (1975).
9. H. P. A. H. Irwin and P. A. Smith, Prediction of the effect of streamline curvature on turbulence, *Physics Fluids* **18**(6), 624–630 (1975).
10. B. E. Launder, G. J. Reece and W. Rodi, Progress in the development of a Reynolds-stress turbulence closure, *J. Fluid Mech.* **68**(3), 537–566 (1975).
11. M. M. Gibson and B. E. Launder, On the calculation of horizontal, turbulent, free shear flows under gravitational influence, *J. Heat Transfer* **98C**, 81–87 (1976).
12. M. M. Gibson and B. E. Launder, Ground effects on pressure fluctuations in the atmospheric boundary layer, *J. Fluid Mech.* **86**(3), 491 (1978).
13. C. C. Shir, A preliminary numerical study of atmospheric turbulent flows in the idealised planetary boundary layer, *J. Atmos. Sci.* **30**, 1327–1339 (1973).

14. W. Rodi, A new algebraic relation for calculating the Reynolds stresses, *Z. Angew. Math. Mech.* **56**, 219–221 (1976).
15. J. H. Champagne, V. G. Harris and S. Corrsin, Experiments on nearly homogeneous shear flow, *J. Fluid Mech.* **41**(1), 81–141 (1970).
16. B. E. Launder, Heat and mass transport, in *Turbulence*, Chapter 6, edited by Bradshaw. Springer, Berlin (1976).
17. R. M. So and G. L. Mellor, Experiment on convex curvature effects in turbulent boundary layers, *J. Fluid Mech.* **60**(1), 43–62 (1973).
18. S. P. S. Arya and E. J. Plate, Modelling of the stably stratified atmospheric boundary layer, *J. Atmos. Sci.* **26**, 656–665 (1969).
19. N. E. Busch, On the mechanics of atmospheric turbulence, in *Workshop on Micrometeorology*, edited by Haugen. pp. 1–65, Am. Met. Soc. (1972).
20. H. Thomann, Effect of streamwise wall curvature on heat transfer in a turbulent boundary layer, *J. Fluid Mech.* **33**(2), 283–292 (1968).

MODELE ALGEBRIQUE DE TENSION ET DE FLUX THERMIQUE POUR UN ECOULEMENT TURBULENT AVEC COURBURE DES LIGNES DE COURANT

Résumé—Les équations de transport des tensions de Reynolds et du flux thermique pour un écoulement turbulent avec courbure des lignes de courant sont fermées en explicitant les termes de redistribution de façon à traduire la modification du champ de pression fluctuante par la présence de la paroi. Un système d'équations algébriques est établi pour des couches minces avec courbure modérée, pour déduire les effets de courbure sur la longueur de mélange et le nombre de Prandtl turbulent. Des calculs montrent que le transfert thermique est sensiblement moins affecté par la courbure de la ligne de courant que par la tension tangentielle.

EIN ALGEBRAISCHES SPANNUNGS- UND WÄRMESTROM-MODELL FÜR TURBULENTE SCHERSTRÖMUNG BEI GEKRÜMMTEN STROMLINIEN

Zusammenfassung—Die Transportgleichungen für Wandschubspannung und Wärmestrom in turbulenter Strömung mit gekrümmten Stromlinien werden geschlossen dadurch formuliert, daß die redistributiven Terme in einer Weise modelliert werden, welche die Veränderungen des fluktuierenden Druckfeldes in Gegenwart einer Wand wiedergibt. Ein Satz algebraischer Gleichungen wird für dünne Grenzschichten mit mäßiger Krümmung aufgestellt, aus denen der Einfluß der Krümmung auf den Mischungsweg und die turbulente Prandtl-Zahl abgeleitet wird. Rechnungen zeigen, daß der Wärmeübergang von der Stromlinienkrümmung wesentlich weniger beeinflußt wird als die Schubspannung.

АЛГЕБРАИЧЕСКАЯ МОДЕЛЬ НАПРЯЖЕНИЯ И ТЕПЛООТДАЧИ ДЛЯ ТУРБУЛЕНТНОГО ПОТОКА ВЯЗКОЙ ЖИДКОСТИ НА КРИВОЛИНЕЙНОЙ ПОВЕРХНОСТИ

Аннотация — Уравнения переноса для напряжений Рейнольдса и теплового потока при турбулентном течении на криволинейной поверхности замыкаются с помощью модели, учитывающей влияние стенки на изменение пульсационного поля давления. Выведена система алгебраических уравнений для описания пограничных слоёв небольшой толщины и умеренной кривизны, из которых можно определить влияние кривизны на длину пути смешения и турбулентное число Прандтля. Расчёты показывают, что по сравнению с касательным напряжением теплообмен значительно меньше зависит от криволинейности обтекаемой поверхности.